

# Descent data of cosimplicial 2-groupoids

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## Abstract

We interpret descent data of a cosimplicial 2-groupoid as a 2-groupoid which in turn is the homotopy limit of the cosimplicial simplicial set gotten after applying the 2-nerve in each cosimplicial degree. As a corollary, we generalize a result of Yekutieli and prove: if  $\mathcal{G} \rightarrow \mathcal{H}$  is a weak equivalence of cosimplicial 2-groupoids, the map  $\mathbf{Desc}(\mathcal{G}) \rightarrow \mathbf{Desc}(\mathcal{H})$  is a weak equivalence of 2-groupoids.

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## 1 Introduction

In this note we reinterpret algebro-geometric information, namely descent data, in a homotopically-invariant way. Given a cosimplicial 2-groupoid  $\mathcal{G}$ , its descent data is a 2-groupoid  $\mathbf{Desc}(\mathcal{G})$  (or rather its 2-nerve) whose path components coincide with what is called in [Ye1, Definitions 1.4,1.5] the set of descent data quotient out by the gauge equivalence relation. We show that this 2-groupoid is (canonically equivalent to) the homotopy limit  $\mathrm{holim}_{\Delta} \mathbf{N}\mathcal{G}$  where  $\mathbf{N}$  is the 2-nerve applied on each level. Thus, given a weak equivalence of cosimplicial 2-groupoids  $\mathcal{G} \rightarrow \mathcal{H}$ , the map  $\mathbf{Desc}(\mathcal{G}) \rightarrow \mathbf{Desc}(\mathcal{H})$  is a weak equivalence of 2-groupoids. This generalizes [Ye1, Theorem 0.1] and similar to the case of cosimplicial 1-groupoids as in [Jar, Lemma 12].

We know of two situations in which this setup can arise.

The first concerns Maurer-Cartan equations. Consider a cosimplicial DGLA, which shows up for instance as the Čech construction for a sheaf of DGLAs. Taking the Deligne 2-groupoid (which encodes solutions to Maurer-Cartan equations) of each cosimplicial degree gives rise to a cosimplicial 2-groupoid. As follows from [Ye2, Theorem 0.4], a quasi-isomorphism of cosimplicial pronilpotent DGLAs of quantum type (i.e. concentrated in degrees  $\geq -1$ ) induces a weak equivalence of cosimplicial 2-groupoids.

The second is in the classification of  $\mathcal{G}$ -gerbes for a sheaf of groups  $\mathcal{G}$  (see [Br1], [Br2]). There, the cosimplicial 2-groupoid arises via the Čech construction (with respect to a cover) from the sheaf of crossed modules (or 2-groups)  $\mathcal{G} \rightarrow \mathrm{Aut}(\mathcal{G})$  and the descent data approximates isomorphism classes of  $\mathcal{G}$ -gerbes. In some cases, for example when the cover totally trivializes all  $\mathcal{G}$ -gerbes,

$\pi_0 \mathbf{Desc}(\mathcal{G})$  will classify all  $\mathcal{G}$ -gerbes and a refinement will yield a weak equivalence of cosimplicial 2-groups.

Descent data is intimately related to non-abelian cohomology. For this reason, the role of codegeneracies is degenerate and we can consider the restricted totalization (see §2) which simplifies the homotopical framework. This removes the difficulty arising from the fact that the cosimplicial simplicial set gotten by taking the 2-nerve of each level of a cosimplicial 2-groupoid need not be Reedy fibrant (see [Jar, Example 8]). Although it is not necessary for this note we believe that a similar argument to that of [Jar] can be given to prove that for a cosimplicial 2-groupoid  $\mathcal{G}$ ,  $Tot(\mathbf{NG}) \simeq \text{holim}_{\Delta} \mathbf{NG}$ .

The simplicity of the argument made here makes it valid in a much more general setting, namely for cosimplicial weak n-groupoids. This is discussed in §7.

## 2 Totalization and Restricted Totalization

Let  $\Delta$  be the category whose objects are non-empty finite ordinals  $[0], [1], \dots, [n]$  where  $[n] = \{0, 1, \dots, n\}$  and whose morphisms are weakly order preserving functions. Every morphism in  $\Delta$  is a composition of face maps  $\delta^i : [n-1] \rightarrow [n]$  and degeneracies  $\sigma^i : [n+1] \rightarrow [n]$ ,  $i = 0, \dots, n$ . A simplicial set is a functor  $X : \Delta^{op} \rightarrow \mathbf{Set}$  and we write  $X_n := X([n])$ ,  $d_i := X(\delta^i)$ ,  $s_i := X(\sigma^i)$ . Write  $s\mathbf{Set}$  for the category whose objects are simplicial sets and whose morphisms are natural transformations.

A *cosimplicial object* in a category  $\mathcal{C}$  is a functor  $\Delta \rightarrow \mathcal{C}$ . In particular, a *cosimplicial simplicial set* is a cosimplicial object in  $s\mathbf{Set}$ . We write  $s\mathbf{Set}^{\Delta}$  for the category whose objects are cosimplicial simplicial sets and whose morphisms are natural transformations. If  $X$  is a cosimplicial object we will denote the object assigned to  $[n]$  by  $X^n$ . The maps  $d^i := X(\delta^i)$  and  $s^i := X(\sigma^i)$  will be called cofaces and codegeneracies respectively. The *cosimplicial standard simplex*  $\Delta$  has  $\Delta^n$  as its n-th level and the obvious coface and codegeneracy maps. The category  $s\mathbf{Set}^{\Delta}$  is enriched over simplicial sets. Given  $X, Y \in s\mathbf{Set}^{\Delta}$ , the ‘internal Hom’  $\underline{Hom}(X, Y)$  is the simplicial set whose n-simplices are

$$\underline{Hom}(X, Y)_n = Hom_{s\mathbf{Set}^{\Delta}}(X \times \Delta^n, Y)$$

Here,  $X \times \Delta^n$  is the product of  $X$  with the constant cosimplicial simplicial set  $\Delta^n$ .

**Definition 2.1.** The *totalization*  $Tot : s\mathbf{Set}^{\Delta} \rightarrow s\mathbf{Set}$  is the simplicial set  $Tot(X) = \underline{Hom}(\Delta, X)$ .

We let  $\Delta_{res}$  denote the subcategory of  $\Delta$  with the same objects but only face (i.e. injective) maps  $\delta^i$ . A *restricted cosimplicial object* in a category  $\mathcal{C}$  is a functor  $\Delta_{res} \rightarrow \mathcal{C}$ ; it is also called a *semi-cosimplicial object* by some authors. In particular, a restricted cosimplicial object in  $s\mathbf{Set}$  is called a *restricted cosimplicial*

*simplicial set*. There is an obvious ‘restriction’ functor  $r : sSet^{\Delta} \rightarrow sSet^{\Delta_{res}}$  and in particular we have  $r\Delta \in sSet^{\Delta_{res}}$ . The category  $sSet^{\Delta_{res}}$  is again enriched over simplicial sets. If  $X, Y \in sSet^{\Delta_{res}}$  we have  $\underline{Hom}_r(X, Y) \in sSet$ . Its  $n$ -simplices are  $\underline{Hom}_r(X, Y)_n := Hom_{sSet^{\Delta_{res}}}(X \times r\Delta^n, Y)$  and given  $\theta : [m] \rightarrow [n]$  in  $\Delta$ , the map  $\theta^* : \underline{Hom}_r(X, Y)_n \rightarrow \underline{Hom}_r(X, Y)_m$  is induced by composing with the map  $\theta_* : r\Delta^m \rightarrow r\Delta^n$ . Simplicial identities hold since their opposites hold in  $\Delta$ . To avoid notation overload, we will omit the subscript ‘r’ and write  $\underline{Hom}(X, Y)$  for the internal hom in  $sSet^{\Delta_{res}}$ .

**Definition 2.2.** The *restricted totalization* is the functor  $Tot_r : sSet^{\Delta_{res}} \rightarrow sSet$  defined by  $Tot_r(X) = \underline{Hom}(r\Delta, X)$ .

### 3 Model structures

We assume the reader is familiar with the definition of a model category. Let us shortly spell out the definition of a simplicial model category.

**Definition 3.1.** A model category  $\mathcal{M}$  is called *simplicial* if it is enriched with tensor and cotensor over  $sSet$  and satisfies the following axiom [Qui, II.2 SM7]: If  $f : A \rightarrow B$  is a cofibration in  $\mathcal{M}$  and  $i : K \rightarrow L$  is a cofibration in  $sSet$  then the map

$$q : A \otimes L \coprod_{A \otimes K} B \otimes K \rightarrow B \otimes L$$

1. is a cofibration;
2. is a weak equivalence if either
  - (a)  $f$  is a weak equivalence in  $\mathcal{M}$  or
  - (b)  $i$  is a weak equivalence in  $sSet$ .

**Definition 3.2.** A category  $\mathcal{R}$  is called a *Reedy category* if it has two subcategories  $\mathcal{R}_+, \mathcal{R}_- \subseteq \mathcal{R}$  and a *degree* function  $d : ob(\mathcal{R}) \rightarrow \alpha$  where  $\alpha$  is an ordinal number such that:

- Every nonidentity morphism in  $\mathcal{R}_+$  raises degree;
- Every nonidentity morphism in  $\mathcal{R}_-$  lowers degree;
- Every morphism in  $\mathcal{R}$  factors uniquely as a map in  $\mathcal{R}_-$  followed by a map in  $\mathcal{R}_+$ .

The category  $\Delta$  is a Reedy category with  $\Delta_+ = \Delta_{inj}$  ( $= \Delta_{res}$ ),  $\Delta_- = \Delta_{surj}$  and the obvious degree function.

Let  $\mathcal{R}$  be a Reedy category and  $\mathcal{C}$  any category. Given a functor  $X : \mathcal{R} \rightarrow \mathcal{C}$  and an object  $n \in \mathcal{R}$ , define the  $n$ -th *latching object* to be

$$L_n X = colim_{\mathbb{L}(\mathcal{R})} X_s$$

where  $\mathbb{L}(\mathcal{R})$  the full subcategory of the over category  $\mathcal{R}_+/n$  containing all objects except the identity  $id_n$ .

Dually, define the  $n$ -th *matching object* to be

$$M_n X = \lim_{\mathbb{M}(\mathcal{R})} X_s$$

where  $\mathbb{M}(\mathcal{R})$  is the full subcategory of the under category  $n/\mathcal{R}_-$  containing all objects except  $id_n$ . There are natural morphisms

$$L_n X \rightarrow X_n \rightarrow M_n X.$$

The importance of a Reedy structure on  $\mathcal{R}$  is due to the following:

**Theorem 3.3.** *[Re],[Ang, Theorem 4.7] Let  $\mathcal{R}$  be a Reedy category and  $\mathcal{M}$  a model category. The functor category  $\mathcal{M}^{\mathcal{R}}$  admits a structure of a model category, called Reedy model structure in which a map  $X \rightarrow Y$  is a*

- *Weak equivalence iff  $X_n \rightarrow Y_n$  is a weak equivalence in  $\mathcal{M}$  for every  $n$ .*
- *Cofibration iff the map  $L_n Y \amalg_{L_n X} X_n \rightarrow Y_n$  is a cofibration in  $\mathcal{M}$  for every  $n$ .*
- *Fibration iff the map  $X_n \rightarrow M_n X \times_{M_n Y} Y_n$  is a fibration in  $\mathcal{M}$  for every  $n$ .*

In particular, an object  $X$  is

- *Fibrant iff  $X_n \rightarrow M_n X$  is a fibration in  $\mathcal{M}$  for every  $n$ .*
- *Cofibrant iff  $L_n X \rightarrow X_n$  is a cofibration in  $\mathcal{M}$  for every  $n$ .*

Moreover, if the model structure on  $\mathcal{M}$  is simplicial, so is the Reedy model structure on  $\mathcal{M}^{\mathcal{R}}$ .

**Corollary 3.4.** *The Quillen model structure  $sSet_{Quillen}$  and the Reedy structure on  $\Delta$  (respectively  $\Delta_{res}$ ) induce a simplicial model structure on  $sSet^{\Delta}$  (respectively  $sSet^{\Delta_{res}}$ ), denoted  $sSet_{Reedy}^{\Delta}$ .*

**Example 1.** The object  $X = \Delta \in sSet^{\Delta}$  is Reedy cofibrant. The map  $L_n X \rightarrow X^n$  is the inclusion  $\partial \Delta^n \hookrightarrow \Delta^n$  which is a cofibration of simplicial sets.

Next, we recall another model structure on  $sSet^{\Delta_{res}}$ .

**Theorem 3.5.** *The simplicial enrichment of  $sSet^{\Delta_{res}}$  can be extended to a simplicial model structure, called the projective model structure, in which a map  $X \rightarrow Y$  is a*

- *Weak equivalence iff for each  $n$ ,  $X^n \rightarrow Y^n$  is a weak equivalence.*
- *Fibration iff for each  $n$ ,  $X^n \rightarrow Y^n$  is a Kan fibration.*

- *Cofibration* iff it has the left lifting property with respect to trivial fibrations.

In particular,  $X$  is a fibrant object iff  $X^n$  is a Kan complex for every  $n$ .

Suppose  $\mathcal{R}$  is a Reedy category and  $\mathcal{M}$  is a model category. In general, if the projective model structure on  $\mathcal{M}^{\mathcal{R}}$  exists (e.g. when  $\mathcal{M}$  is sufficiently nice) it will be very different than the Reedy model structure. However, in some cases the two may coincide:

**Theorem 3.6.** (cf. [Lur, A.2.9.22]) *If  $\mathcal{R} = \mathcal{R}_+$  the projective and Reedy model structures on  $\mathcal{M}^{\mathcal{R}}$  coincide.*

In the case  $\mathcal{R} = \Delta_{res}$  we obtain:

**Corollary 3.7.** *The Reedy and projective model structures on  $sSet^{\Delta_{res}}$  coincide.*

Thus, an object  $X \in sSet^{\Delta_{res}}$  is Reedy fibrant iff  $X^n$  is a Kan complex for each  $n$ .

By example 1,  $\Delta$  is Reedy cofibrant in  $sSet^{\Delta}$  and since the indexing category defining  $L_n\Delta$  depends only on  $\Delta_+ = \Delta_{res}$ , we have  $L_n\Delta = L_nr\Delta$ . Thus, the map  $L_nr\Delta \rightarrow r\Delta^n$  is again the inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$  so that  $r\Delta$  is Reedy cofibrant in  $sSet^{\Delta_{res}}$ .

## 4 2-Groupoids

**Definition 4.1.** A (strict) 2-groupoid is a groupoid-enriched (small) category in which all morphisms are invertible.

Explicitly it is the data  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2)$  consisting of

- A set of objects  $\mathcal{G}_0$ .
- A collection of sets  $\mathcal{G}_1 = \{\mathcal{G}_1(x, y)\}$  indexed by pairs of objects  $x, y \in \mathcal{G}_0$ . Elements of  $\mathcal{G}_1(x, y)$  are called *1-morphisms*. We write  $f : x \rightarrow y$  for  $f \in \mathcal{G}_1(x, y)$ .
- A collection of sets  $\mathcal{G}_2 = \{\mathcal{G}_2(x, y)(f, g)\}$  indexed by pairs of objects  $x, y \in \mathcal{G}_0$  and by pairs of 1-morphisms  $f, g \in \mathcal{G}_1(x, y)$ . The elements of  $\mathcal{G}_2(x, y)(f, g)$  are called *2-morphisms*. We write  $\alpha : f \Rightarrow g$  for  $\alpha \in \mathcal{G}_2(x, y)(f, g)$ .
- A composition law

$$\circ_1 : \mathcal{G}_1(x, y) \times \mathcal{G}_1(y, z) \rightarrow \mathcal{G}_1(x, z)$$

called *horizontal composition of 1-morphisms* and denoted  $\circ$  by abuse.

- A composition law

$$\circ_2 : \mathcal{G}_2(x, y)(f, f') \times \mathcal{G}_2(y, z)(g, g') \rightarrow \mathcal{G}_2(x, z)(g \circ f, g' \circ f')$$

called *horizontal composition for 2-morphisms* and denoted  $\circ$  by abuse.

- A composition law

$$* : \mathcal{G}_2(x, y)(f, g) \times \mathcal{G}_2(x, y)(g, h) \rightarrow \mathcal{G}_2(x, y)(f, h)$$

called *vertical composition for 2-morphisms*.

- A 1-morphism  $1_x \in \mathcal{G}_1(x, x)$  for every object  $x \in \mathcal{G}_0$ .
- A 2-morphism  $1_f \in \mathcal{G}_2(x, y)(f, f)$  for every  $f \in \mathcal{G}_1(x, y)$ .

The conditions are:

- The pair  $(\mathcal{G}_0, \mathcal{G}_1)$  is a groupoid with set of objects  $\mathcal{G}_0$ , sets of morphisms  $\mathcal{G}_1(x, y)_{x, y \in \mathcal{G}_0}$  composition law  $\circ_1$  and identity morphisms  $1_x$ .
- For every  $x, y \in \mathcal{G}_0$ , the pair  $(\mathcal{G}_1(x, y), \mathcal{G}_2(x, y))$  is a groupoid with set of objects  $\mathcal{G}_1(x, y)$  sets of morphisms  $\mathcal{G}_2(x, y)(f, g)_{f, g \in \mathcal{G}_1(x, y)}$  composition law  $*$  and identity morphisms  $1_f$ .
- The composition  $\circ_2$  is associative and compatible with  $*$  via the interchange law

$$(\beta * \alpha) \circ (\beta' * \alpha') = (\beta' \circ \beta) * (\alpha' \circ \alpha)$$

*Remark 4.2.* Note that it follows from the interchange law that 2-morphisms are invertible with respect to horizontal composition as well.

We denote by  $2\mathcal{Gpd}$  the category of 2-groupoids and strict 2-functors between them. The following is a well-known observation.

**Theorem 4.3.** *There is a natural 3-category structure on  $2\mathcal{Gpd}$  with 2-morphisms the 2-natural transformations and 3-morphisms the modifications.*

Let  $\Delta_{\leq n}$  be the full subcategory of  $\Delta$  with objects  $[0], \dots, [n]$  and let  $sSet_{\leq n}$  be the category of functors  $(\Delta_{\leq n})^{op} \rightarrow Set$ . Objects of  $sSet_{\leq n}$  are called *n-truncated simplicial sets*. The inclusion  $\Delta_{\leq n} \rightarrow \Delta$  induces a ‘truncation functor’  $tr_n : sSet \rightarrow sSet_{\leq n}$  which admits right and left adjoints  $cosk_n : sSet_{\leq n} \rightarrow sSet$  and  $sk_n : sSet_{\leq n} \rightarrow sSet$  respectively. We denote by  $Cosk_n : sSet \rightarrow sSet$  the composition  $cosk_n \circ tr_n$  and by  $Sk_n$  the composition  $sk_n \circ tr_n$ . The functor  $Sk_n$  takes a simplicial set and creates a new simplicial set from its n-truncation by adding degenerate simplices in all levels above  $n$ ; it is the simplicial analogue of the n-skeleton of a CW complex. The functor  $Cosk_n$  has a more involved simplicial description; it is the simplicial analogue of the  $(n - 1)$ th Postnikov piece  $P_{n-1}A$  of a space  $A$ .

It follows from formal considerations that  $Cosk_n$  is right adjoint to  $Sk_n$ . Thus, maps  $X \rightarrow Cosk_n Y$  are in 1-1 correspondence with maps  $Sk_n X \rightarrow Y$

**Definition 4.4.** A simplicial set  $X$  is called *n-coskeletal* if the canonical map  $X \rightarrow \text{Cosk}_n X$  is an isomorphism.

In particular, given an  $n$ -truncated simplicial set  $X$ ,  $\text{Cosk}_n X$  is an  $n$ -coskeletal simplicial set. Thus, when we wish to define an  $n$ -coskeletal simplicial set we may define only its  $n$ -truncation.

The following is taken from [MS].

**Definition 4.5.** The *2-nerve* is the functor  $\mathbf{N} : 2\mathcal{G}pd \rightarrow s\mathcal{S}et$  which takes a 2-groupoid  $\mathcal{G}$  to the 3-coskeletal simplicial set  $\mathbf{N}\mathcal{G}$  whose

- 0-simplices are  $\mathcal{G}_0$  the objects of  $\mathcal{G}$ .
- 1-simplices are  $\mathcal{G}_1$  the morphisms of  $\mathcal{G}$ .
- 2-simplices are triangles of the form

$$\begin{array}{ccc} & x_2 & \\ g_{02} \nearrow & \alpha & \nwarrow g_{12} \\ x_0 & \xrightarrow{g_{01}} & x_1 \end{array}$$

where  $g_{ij} \in \mathcal{G}_1(x_i, x_j)$  and  $\alpha : g_{02} \Rightarrow g_{12} \circ g_{01}$  a 2-morphism in  $\mathcal{G}_2$ .

- 3-simplices are tetrahedra with faces being 2-simplices as above such that the diagram of 2-morphisms commutes (see [MS, p.9]); we say that the faces of such tetrahedron satisfy the ‘tetrahedron condition’.

We will need three well-known properties of the 2-nerve functor:

**Proposition 4.6.** [MS]

1. For every 2-groupoid  $\mathcal{G}$ ,  $\mathbf{N}\mathcal{G}$  is 3-coskeletal.
2. For every 2-groupoid  $\mathcal{G}$ ,  $\mathbf{N}\mathcal{G}$  is a Kan complex.
3. A map of 2-groupoids  $\mathcal{G} \rightarrow \mathcal{H}$  is a weak equivalence iff  $\mathbf{N}\mathcal{G} \rightarrow \mathbf{N}\mathcal{H}$  is a weak equivalence of simplicial sets.

## 5 Descent data of cosimplicial 2-groupoids

In [Ye1, Definition 1.4], following [BGNT], the author defines descent data for a cosimplicial crossed groupoid. Since crossed groupoids can be viewed as 2-groupoids, a straightforward translation leads to the following definition:

**Definition 5.1.** Given a cosimplicial 2-groupoid  $\mathcal{G} = \{G^n\}$ , a descent datum is a triple  $(x, g, \alpha, )$  such that:

1.  $x \in \mathcal{G}_0^0$

2.  $g \in \mathcal{G}_1^1(d_1x, d_0x)$
3.  $\alpha \in \mathcal{G}_2^2(d_1g, d_0g \circ d_2g)$  ('failure of 1-cocycle').

Which in addition satisfy the 'twisted 2-cocycle' condition.

One can verify that the twisted 2-cocycle condition corresponds to the tetrahedron condition in definition 4.5.

Thus, such triples are in 1-1 correspondence with diagrams of simplicial sets of the form

$$\begin{array}{ccccccc}
 \Delta^0 & \rightrightarrows & \Delta^1 & \rightrightarrows & \Delta^2 & \rightrightarrows & \Delta^3 \\
 x \downarrow & & g \downarrow & & \alpha \downarrow & & t \downarrow \\
 \mathbf{NG}^0 & \rightrightarrows & \mathbf{NG}^1 & \rightrightarrows & \mathbf{NG}^2 & \rightrightarrows & \mathbf{NG}^3
 \end{array}$$

Since  $\mathbf{NG}^n$  is 3-coskeletal, diagrams as above are in turn the 0-simplices

$$Tot_r(r\mathbf{NG})_0 = \underline{Hom}(r\Delta, r\mathbf{NG})_0 = Hom(r\Delta, r\mathbf{NG})$$

Similarly, given two descent data  $(x, g, \alpha), (x', g', \alpha')$ , a gauge transformation between them (see [Yel], definition 1.5) corresponds to a diagram

$$\begin{array}{ccccc}
 \Delta^0 \times \Delta^1 & \rightrightarrows & \Delta^1 \times \Delta^1 & \rightrightarrows & \Delta^2 \times \Delta^1 \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{NG}^0 & \rightrightarrows & \mathbf{NG}^1 & \rightrightarrows & \mathbf{NG}^2
 \end{array}$$

and again since  $\mathbf{NG}^n$  is 3-coskeletal, such diagrams correspond to 1-simplices

$$Tot_r(r\mathbf{NG})_1 = \underline{Hom}(r\Delta, r\mathbf{NG})_1 = Hom(r\Delta \times r\Delta^1, r\mathbf{NG})$$

that restrict to  $(x, g, \alpha)$  and  $(x', g', \alpha')$  via the maps  $d_0, d_1 : \Delta^0 \rightrightarrows \Delta^1$ .

Let  $Desc(\mathcal{G})$  denote the set of descent data of  $\mathcal{G}$  and  $\overline{Desc}(\mathcal{G})$  its quotient by the gauge equivalence relation ([Yel], definition 1.8). We have just proven:

**Theorem 5.2.** *For any cosimplicial 2-groupoid  $\mathcal{G}$ , there is a (natural) isomorphism*

$$\overline{Desc}(\mathcal{G}) \cong \pi_0(Tot_r(r\mathbf{NG}))$$

Let  $\Delta_2$  be the cosimplicial 2-category (i.e. a functor  $\Delta \rightarrow 2Cat$ ) whose values on  $[0], [1]$  and  $[2]$  are correspondingly the 2-categories

$$\begin{array}{c}
 \bullet \\
 \bullet \rightarrow \bullet \\
 \bullet \begin{array}{c} \nearrow \bullet \nwarrow \\ \xrightarrow{\alpha} \end{array} \bullet
 \end{array}$$

and



where  $\alpha$  is a 2-morphism from the left-handed edge to the composition of the other two. All other values of  $\Delta_2$  can similarly be recovered from the description in [MS]. Then the 2-nerve  $\mathbf{N} : 2Cat \rightarrow sSet$ , as defined in [MS], arises from the functor  $\Delta_2$  by setting  $\mathbf{N}(C)_n := ob(Hom_{2Cat}(\Delta_2([n]), C))$ . It follows that the 2-category  $\mathcal{G}^{\Delta_2}$  with objects, 1-morphisms and 2-morphisms being cosimplicial-2-functors, 2-natural transformations and modifications (respectively) admits a structure of a 2-groupoid induced from the one in  $\mathcal{G}$  (see theorem 4.3). In a similar way, the mapping 2-category of restricted cosimplicial 2-categories  $r\mathcal{G}^{r\Delta_2}$  has a structure of a 2-groupoid. As in [Jar] we have

$$(1) \quad Tot(\mathbf{N}\mathcal{G}) \cong \mathbf{N}(\mathcal{G}^{\Delta_2})$$

and similarly

$$(2) \quad Tot_r(\mathbf{N}\mathcal{G}) \cong r\mathbf{N}(\mathcal{G}^{\Delta_2})$$

In light of Theorem 5.2 and the discussion above, we define:

**Definition 5.3.** Given a cosimplicial 2-groupoid  $\mathcal{G}$ , its *descent 2-groupoid* is  $\mathbf{Desc}(\mathcal{G}) := r\mathcal{G}^{r\Delta_2}$ .

## 6 Invariance of descent data

Theorem 5.2 enables us to use homotopy-theoretic tools to prove invariance of descent data under weak equivalence. We need one more simple theorem.

**Theorem 6.1.** *For any cosimplicial 2-groupoid  $\mathcal{G}$ , there is a (natural) weak equivalence  $Tot_r(\mathbf{N}\mathcal{G}) \simeq \text{holim}_{\Delta} \mathbf{N}\mathcal{G}$ ,*

*Proof.* In the simplicial model category  $sSet_{proj}^{\Delta_{res}}$ , the homotopy limit (over  $\Delta_{res}$ ) of a fibrant diagram can be described as the internal mapping space from a weakly contractible cofibrant object. In our case,  $\mathbf{N}\mathcal{G}$  is fibrant and  $r\Delta$  is weakly contractible and cofibrant. Thus,  $Tot_r(r\mathbf{N}\mathcal{G}) = \underline{Hom}(r\Delta, \mathbf{N}\mathcal{G}) \simeq \text{holim}_{\Delta_{res}} r\mathbf{N}\mathcal{G}$ . By ([DF, Lemma 3.8]),  $\text{holim}_{\Delta_{res}} r\mathbf{N}\mathcal{G} \simeq \text{holim}_{\Delta} \mathbf{N}\mathcal{G}$ .  $\square$

It follows that a weak equivalence of cosimplicial 2-groupoids induces a weak equivalence on the restricted totalization. By Proposition 4.6(2) and equation 2 it follows that:

**Corollary 6.2.** *A weak equivalence of cosimplicial 2-groupoids  $\mathcal{G} \rightarrow \mathcal{H}$  induces a weak equivalence of 2-groupoids  $\mathbf{Desc}(\mathcal{G}) \rightarrow \mathbf{Desc}(\mathcal{H})$ .*

In particular, we have:

**Corollary 6.3** (cf. [Ye1], theorem 2.4). *If  $\mathcal{G} \rightarrow \mathcal{H}$  is a weak equivalence of cosimplicial 2-groupoids, the induced map  $\overline{\mathbf{Desc}}(\mathcal{G}) \rightarrow \overline{\mathbf{Desc}}(\mathcal{H})$  is an isomorphism of sets*

*Proof.* By theorems 5.2 and 6.1, the map  $\overline{\mathbf{Desc}}(\mathcal{G}) \rightarrow \overline{\mathbf{Desc}}(\mathcal{H})$  coincides with  $\pi_0(\text{holim}_{\Delta} \mathbf{N}\mathcal{G}) \rightarrow \pi_0(\text{holim}_{\Delta} \mathbf{N}\mathcal{H})$  and  $\mathbf{N}$  and  $\text{holim}_{\Delta}$  preserve weak equivalences.  $\square$

## 7 Descent data of cosimplicial weak $n$ -groupoids

In light of theorem 5.2, it makes sense to define:

**Definition 7.1.** Let  $\mathcal{G}$  be a cosimplicial weak  $n$ -groupoid. Its *descent data* is the simplicial set  $Tot_r(\mathbf{N}\mathcal{G})$  where  $\mathbf{N}$  is the weak  $n$ -nerve.

Definition 7.1 makes sense formally, but its geometric meaning is unknown to us. Nevertheless, theorem 6.1 generalizes immediately.

**Theorem 7.2.** Let  $\mathcal{G}$  be a cosimplicial weak  $n$ -groupoid. There is a natural weak equivalence  $Tot_r(\mathcal{G}) \simeq \mathrm{holim}_{\Delta} \mathbf{N}\mathcal{G}$ . In particular, if  $\mathcal{G} \rightarrow \mathcal{H}$  is a weak equivalence of cosimplicial weak  $n$ -groupoids then the map

$$Tot_r(\mathbf{N}\mathcal{G}) \rightarrow Tot_r(\mathbf{N}\mathcal{H})$$

is a weak equivalence of simplicial sets.

*Proof.* The only ingredient needed here in addition to the proof of Theorem 6.1 is that for each  $n$ ,  $\mathbf{N}\mathcal{G}^n$  is a Kan complex.  $\square$

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